Lp Sampling from Streams

joint work with
Gábor Tardos (Alfréd Rényi Institute of Mathematics)
Hossein Jowhari (MADALGO)

July 25, 2012

Mert Sağlam

Lp Sampling from Update Streams

- The input is an *update stream*.
- We have an *n* dimensional vector *x*, initially zero.
- The input is updates to the coordinates of *x*
- When the stream is exhausted, an ε relative error sampler outputs a coordinate *J* s.t.
- An *augmented sampler* also returns an ε appx. to x_J

x	0	2	0	0	4	-2	0	-2
	1	2	3	4	5	6	7	8

(2,5) (6,-2) (5,4) (2,-3) (8,-2)

$$\Pr[J=i]=(1\pm\epsilon)\frac{|x_i|^p}{\|x\|_p^p}\pm n^{-c}$$

Here, $\|x\|_p^p=\sum_{i=1}^{n}|x_i|^p$

Lp Sampling from Update Streams

- In SODA 2010 Monemizadeh and Woodruff introduced Lp sampling.
- They gave poly($1/\varepsilon$, log n) space ε error Lp samplers for *p* in [0,2].
- In FOCS 2011 Andoni, Krauthgamer and Onak improved space usage to $O(\varepsilon^{-p}\log^4 n)$ bits for *p* in [1,2].
- We give an Lp samplers with $O(\varepsilon^{-p}\log^2 n)$ bits of space for p in [1,2).
- Our sampler works for p in [0,1] too, taking $O(\varepsilon^{-1}\log^2 n)$ space. For p=0 space usage is $O(\log^2 n)$.
- We show that any one pass Lp sampler requires $\Omega(\log^2 n)$ bits.
- Any one pass augmented sampler requires $\Omega(\varepsilon^{-p}\log n)$ space.

Our Lp Sampler for p=1

- The bare-bones algorithm
- For i=1,…,n pick r_i uniformly at random from real interval [0,1]
- Set $z_i = x_i / r_i$.
- Find i with $|z_i|$ maximal.
- If $|z_i| > \varepsilon^{-1}\|x\|_1$, output $J=i$, otherwise output FAIL.

x	0	2	0	0	4	-2	0	-2
/								
r	0.3	0.2	0.4	0.9	0.2	0.4	0.2	0.1
=								
z	0	10	0	0	20	-5	0	-20

What is the probability that we output coordinate i ?

Our Lp Sampler for p=1

Claim 1: $\Pr[J = i] \le \varepsilon |x_i| / \|x\|_1$

- We output a coordinate only if $|z_i| > \varepsilon^{-1}\|x\|_1$.
- This happens only when $|x_i| / r_i > \varepsilon^{-1}\|x\|_1$.

Claim 2: $\Pr[J=i] \ge (\varepsilon-\varepsilon^2)|x_i| / \|x\|_1$

- Conditioned on $|z_i| > \varepsilon^{-1}\|x\|_1$, probability that $|z_j| > \varepsilon^{-1}\|x\|_1$ is $\le \varepsilon |x_j| / \|x\|_1$ by Claim 1.
- Union bound over all j, $\exists j$ has probability ε.

x	0	2	0	0	4	-2	0	-2
/								
r	0.3	0.2	0.4	0.9	0.2	0.4	0.2	0.1
=								
z	0	10	0	0	20	-5	0	-20

Our Lp Sampler for p=1

- By Claim 2, $\Pr[J=i] \geq (\varepsilon-\varepsilon^2)\, |x_i|/\|x\|_1$
- Summing over all j, we see that the procedure outputs a coordinate with probability $(\varepsilon-\varepsilon^2)$
- Hence if we repeat in parallel $O(\varepsilon^{-1}\log(1/\delta))$ times, and return the first non failing output, we get a coordinate with $(1-\delta)$ probability.

x	0	2	0	0	4	-2	0	-2
/								
r	0.3	0.2	0.4	0.9	0.2	0.4	0.2	0.1
=								
z	0	10	0	0	20	-5	0	-20

But how do we find max coordinate of z in small space ?

We don't..

Our Lp Sampler for p=1

- Take O(log n) random binary strings $m^1,\dots,m^{\log n}$ each of length n
- Take O(log n) n dimensional random +-1 vectors $k^1,\dots$
- Calculate $z * m^l * k^l$ for $l=1,\dots,\log n$. Here $*$ is coordinate-wise multiplication.
- Estimate z_i by the median of $z_i \times m^l_i \times k^l_i$ for all l.

x	0	2	0	0	4	-2	0	-2
/								
r	0.3	0.2	0.4	0.9	0.2	0.4	0.2	0.1
=								
z	0	10	0	0	20	-5	0	-20

- It is known that $z^*_i = z_i \pm 3\|z\|_2$ with all but n^{-c} probability.

Our Lp Sampler for p=1

- Approximating z_i by z^*_i changes our analysis only if

$$\epsilon^{-1}\|x\|_1 - \|z\|_2 \leqslant |z_i| \leqslant \epsilon^{-1}\|x\|_1 + \|z\|_2$$

- Conditioned on $\|z\|_2 < 10\|x\|_1$, z_i is in this interval only with probability $2\varepsilon^2|x_i|/\|x\|_1$

- Condition $\|z\|_2 < 10\|x\|_1$ happens with good probability and can be detected if does not happen via standard norm estimation algorithms.

x	0	2	0	0	4	-2	0	-2
/								
r	0.3	0.2	0.4	0.9	0.2	0.4	0.2	0.1
=								
z	0	10	0	0	20	-5	0	-20

Finding Duplicates

- Given an array of length n+1 where each item is in [1..n] find an item that appears at least twice.
- By pigeonhole principle a duplicate exists.
- There is a O(1) words RAM algorithm due to Floyd that runs in linear time.
- In the streaming model, a folklore p pass deterministic algorithm with $O(n^{1/p} \log^{1-1/p})$ space.

A	5	1	2	7	2	4	3	6

Finding Duplicates

- Muthukrishnan asks whether there exists a constant pass polylog space algorithm.
- In 2007, Tarui shows that any deterministic p pass algorithm needs $\Omega(n^{1/(2p-1)})$ space.
- In SODA'09 Gopalan and Radhakrishnan give a one pass $O(\log^3 n)$ space randomized algorithm.

A	5	1	2	7	2	4	3	6

- We give a $O(\log^2 n)$ space one pass randomized algorithm
- We show that any one pass algorithm takes $\Omega(\log^2 n)$ space.

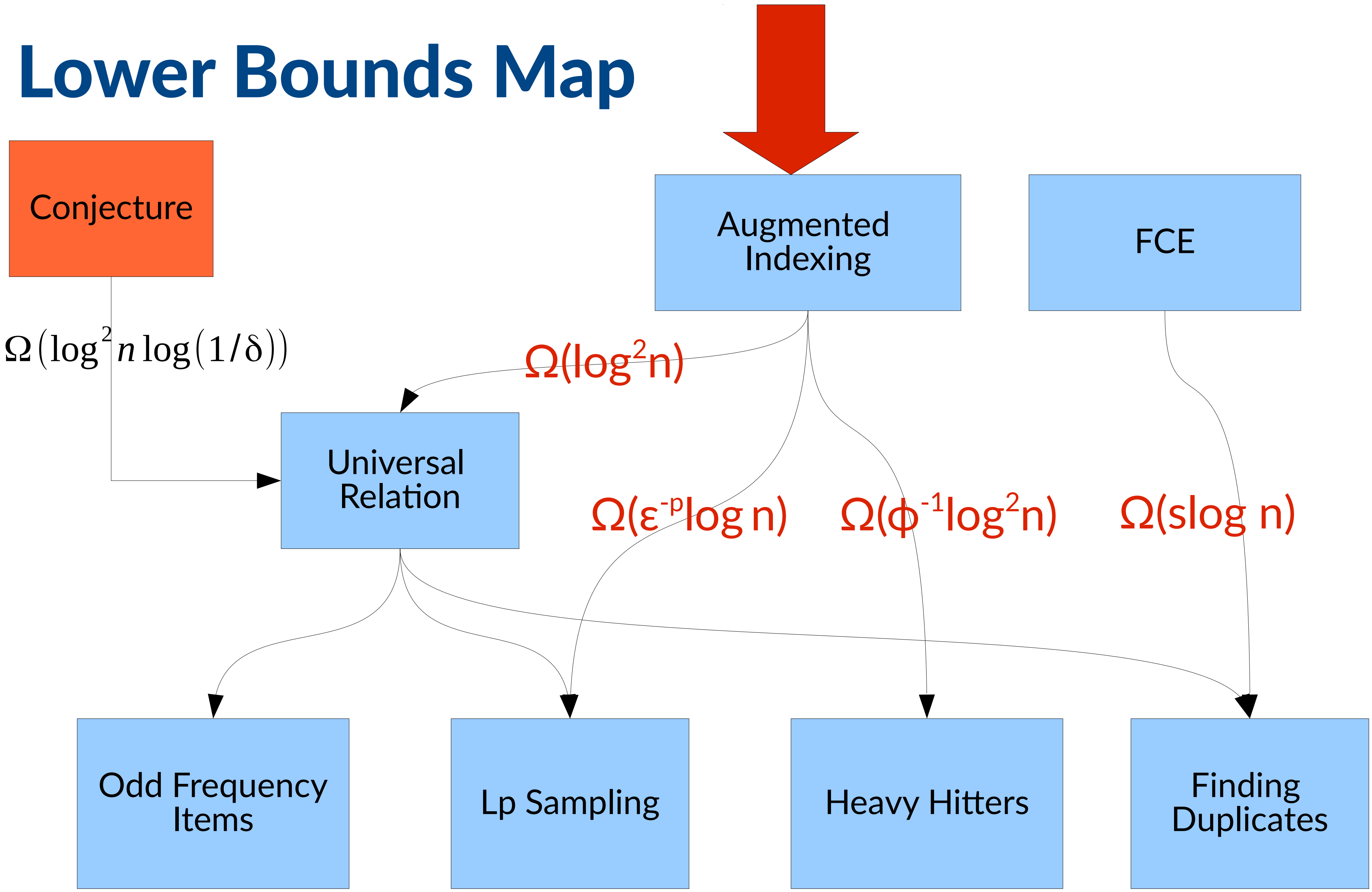
Finding Duplicates Upper Bound

- Run the ½ relative error sampler on a vector x.
- Subtract 1 from each coordinate of x.
- For each item *i* increment x_i by one.
- For each item *i* that appears multiple times, x_i>0.
- We have n decrements and n+1 increments.

A	5	1	2	7	2	4	3	6

- Hence a perfect L1 sample returns a positive coordinate with more than ½ probability.
- ½ relative error sampler returns positive coordinate with constant probability.
- We run O(log(1/delta)) instances of the L1 sampler and return the first positive coordinate.

Lower Bounds Map

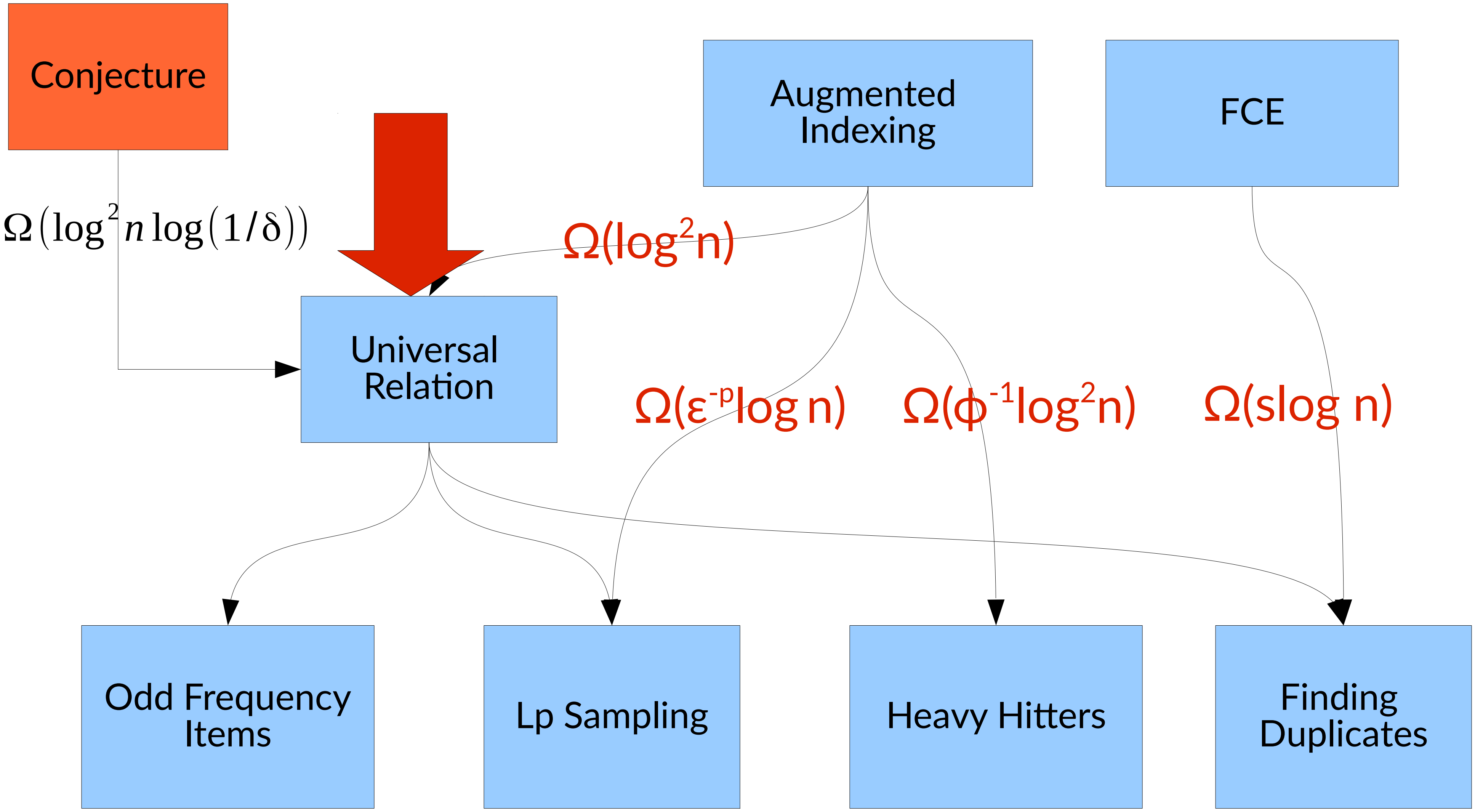


Augmented Indexing Problem

- Alice is given a length n string x over the alphabet [m].
- Alice sends a single message to Bob.
- Bob is given $i \in [n]$ and x_j for $j < i$.
- Bob's goal is to output x_i.

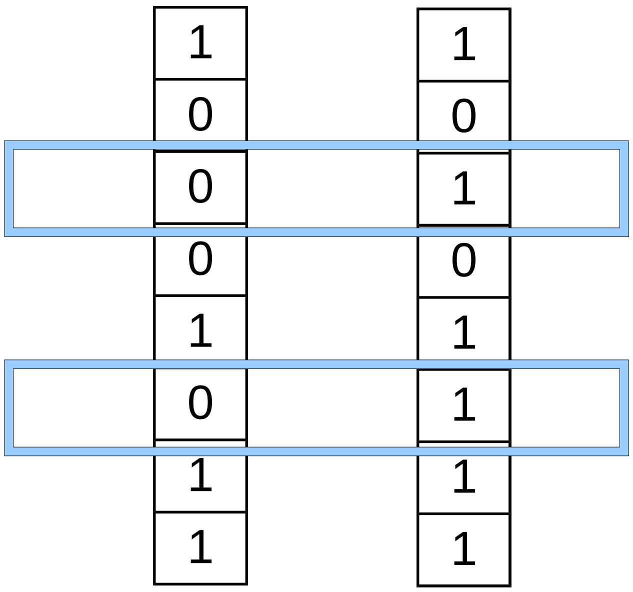
We show that in any one round protocol with $(1-\delta)$ success probability, Alice sends a message of size $\Omega(n \log m)$ whenever $(1-\delta) > 1/m^{1-\varepsilon}$

Lower Bounds Map



Universal Relation

- Alice and Bob are given a binary string each.
- Call these strings x and y.
- Players exchange messages and the last player outputs a coordinate i such that $x_i \neq y_i$.



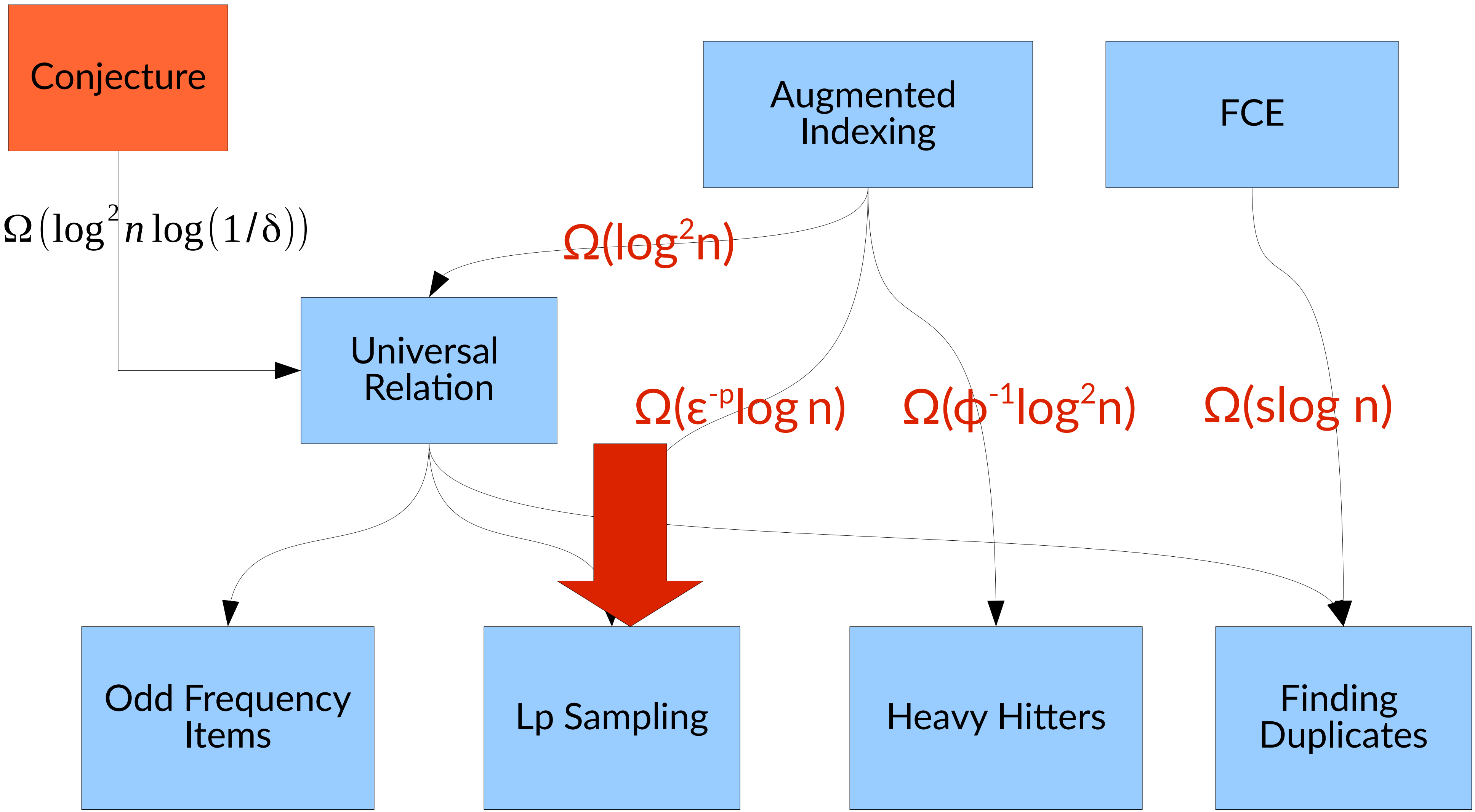
Universal Relation

- Suppose Alice get a length s string z over $[2^t]$.
- Bob gets $i \in [s]$ and z_j for $j < i$.
- The players construct vectors u and v as follows.
- Let e_i be the 2^t dimensional vector 0 everywhere except coordinate i and is 1 in coordinate i.
- For $j=1,\ldots,s$ Alice appends 2^{s-j} copies of e_{z_j}. This is u.
- For $j=1,\ldots,i-1$ Bob appends 2^{s-j} copies of e_{z_j}. Bob appends zeros to reach length $|u|$. This is v.
- They randomly shuffle the positions in u and v.
- A mismatch reveals x_i with ½ probability.

Universal Relation

- Setting $s = t = O(\log n)$ guarantees that $|u| = |v| < n$
- By the augmented indexing lower bound we have $\Omega(st)=\Omega(\log^2 n)$ lower bound.

Lower Bounds Map



Lp Sampling Lower Bound

- Alice and Bob are given binary strings u and v.
- Suppose there is a one pass Lp sampler with S space.
- We give a one round universal relation protocol that communicates S bits.
- Let x be the vector the sampling algorithm implicitly keeps.
- Alice generates updates so that x = u.
- Bob generates updates so that x = u -v.
- We see that x_i is positive iff $u_i \neq v_i$.
- Any Lp sampler returns a positive coordinate with constant probability. Hence an $\Omega(\log^2 n)$ lower bound holds.

Thank You!

Questions?